

# RATIONAL SYMPLECTIC COORDINATES ON THE SPACE OF FUCHS EQUATIONS $m \times m$ -CASE

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**ABSTRACT.** A method of constructing of Darboux coordinates on a space that is a symplectic reduction with respect to a diagonal action of  $\mathrm{GL}(m, \mathbb{C})$  on a Cartesian product of  $N$  orbits of coadjoint representation of  $\mathrm{GL}(m, \mathbb{C})$  is presented. The method gives an explicit symplectic birational isomorphism between the reduced space on the one hand and a Cartesian product of  $N - 3$  coadjoint orbits of dimension  $m(m - 1)$  on an orbit of dimension  $(m - 1)(m - 2)$  on the other hand. In a generic case of the diagonalizable matrices it gives just the isomorphism that is birational and symplectic between some open, in a Zariski topology, domain of the reduced space and the Cartesian product of the orbits in question.

The method is based on a Gauss decomposition of a matrix on a product of upper-triangular, lower-triangular and diagonal matrices. It works uniformly for the orbits formed by diagonalizable or not-diagonalizable matrices. It is elaborated for the orbits of maximal dimension that is  $m(m - 1)$ .

## 0. INTRODUCTION

A set of Fuchs equations<sup>1</sup>

$$\frac{d}{d\lambda}\Psi = \sum_{n=1}^N \frac{A^{(n)}}{\lambda - \lambda_n} \Psi, \quad \Psi \in \mathrm{GL}(m), \quad A^{(n)} \in \mathfrak{gl}(m)$$

may be considered as a submanifold  $\sum_{n=1}^N A^{(n)} = 0$  of the space  $\mathfrak{gl}(m) \times \cdots \times \mathfrak{gl}(m)$  that has a natural Poisson structure coming from  $\mathfrak{gl}(m)$ .

Some important problems like a problem of isomonodromic deformations may be restricted on a “symplectic leaf” of this Poisson manifold, that is a submanifold on which the Poisson structure induces the symplectic structure. It is the symplectic structure that we mean in the title.

The symplectic manifold in question is closely related with the well known, “standard” symplectic manifold, the orbit of coadjoint representation of the Lie group. It is well investigated class of manifolds, and the problem of their canonical parametrization dates back to Archimedes who found a symplectomorphism between sphere, that can be considered as an orbit of  $\mathrm{SU}(2)$ , and a circumscribed cylinder.

In the present article we construct an explicit birational symplectic isomorphism between the symplectic leaf of the space of Fuchs equations and a Cartesian product of the coadjoint orbits (for the version of the method for  $2 \times 2$  matrices see [1]).

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*Key words and phrases.* Fuchs equations, isomonodromic deformations, Hamiltonian reduction, rational symplectic coordinates.

<sup>1</sup>Ground field is  $\mathbb{C}$ ,  $\mathrm{GL}(m)$  means  $\mathrm{GL}(m, \mathbb{C})$  etc.

It solves the problem of the canonical parametrization in the sense that any set of canonical coordinates on a coadjoint orbit gives us a set of desired canonical coordinates of the leaf.

The suggested method does not depend on the type of normal Jordan form of matrices  $A^{(n)}$ . The only difference between not-diagonalizable and diagonalizable cases is following.

In the diagonalizable case the method gives the *isomorphism* between some Zariski open domain of the leaf and the Cartesian product of the orbits, but in the non-diagonalizable case it gives the *birational isomorphism* only. The same effect we have in a Painlevé non-diagonalizable case  $2 \times 2$  matrices too. When we parameterize the equations by a degenerate matrix

$$\begin{pmatrix} pq & q \\ -p^2q & -pq \end{pmatrix},$$

we add a divisor  $q = 0, p \in \mathbb{C}$  that does NOT belong to the orbit of  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ , so new divisor that does not belong to the orbit arise.

#### 1. SPACES $\tilde{M}$ , $\tilde{M}|_{\Sigma=0}$ AND $M$ .

Let  $A^{(1)}, \dots, A^{(N)} \in \mathfrak{gl}^N(m)$  be a set of  $N \geq 3$  matrices  $m \times m$ . Let  $\chi_n = \det(\lambda I - A^{(n)})$  be a characteristic polynomial of  $A^{(n)}$ .

**Restriction 1.** *We assume that the matrices  $A^{(n)}$  have the following property: all eigen subspaces of every  $A^{(n)}$  are one-dimensional.*

The Restriction implies that all invariant factors (see [2], another name – *invariant polynomials*, see [3]) except a determinant are equal to unit.

**Note 1.** *A conjugacy class of matrices under the Restriction 1 is well defined by its characteristic polynomial.*

Denote the conjugacy class of matrices with one-dimensional eigenspaces and characteristic polynomial  $\chi$  by  $\mathcal{O}(\chi)$ :

$$\mathcal{O}(\chi) := \bigcup_{g \in \mathrm{GL}(m)} g^{-1} A g, \quad \text{where } \chi := \det(\lambda I - A)$$

**Note 2.** *The restriction implies that  $A^{(n)}$  commutes with polynomials on  $A^{(n)}$  only, and the orbit  $\mathcal{O}(\chi_n)$  is  $m(m-1)$ -dimensional.*

It is well known that  $\mathcal{O}(\chi_n)$  is  $m(m-1)$ -dimensional symplectic space, let us denote it symplectic form by  $\omega_n$ .

Let us consider  $N$  polynomials  $\chi_n$ ,  $n = 1, \dots, N$  of the  $m$ 's order with unit leading coefficients. Let us denote by  $(\tilde{M}, \tilde{\omega}) = (\tilde{M}, \tilde{\omega})_{\chi_1, \dots, \chi_N}$  a symplectic space

$$\mathcal{O}(\chi_1) \times \dots \times \mathcal{O}(\chi_N) =: \tilde{M}, \quad \tilde{\omega} := \omega_1 + \dots + \omega_N.$$

Such a space is parameterized by  $N \times m$  numbers, these are coefficients of  $\chi_n$ ,  $n = 1, \dots, N$ . Elements of  $\tilde{M}$  are the sets  $A^{(1)}, \dots, A^{(N)}$ . We denote such a set by  $A^{(\tilde{n})} : A^{(\tilde{n})} \in \tilde{M}$ .

Let us consider a diagonal action of  $\mathrm{GL}(m)$  on  $\tilde{M}$ :

$$g^{-1} A^{(\tilde{n})} g := g^{-1} A^{(1)} g, \dots, g^{-1} A^{(N)} g \in \tilde{M}, \quad g \in \mathrm{GL}(m).$$

It is well known that it is a Poisson action (see [4], Appendix 5), and a momentum map is

$$A^{(\bar{n})} \longrightarrow \sum_{n=1}^N A^{(n)}.$$

Let us denote a zero level of the momentum by  $\tilde{M}|_{\Sigma=0}$ :

$$A^{(\bar{n})} \in \tilde{M}|_{\Sigma=0} = \mathcal{O}(\chi_1) \times \cdots \times \mathcal{O}(\chi_N)|_{\Sigma=0} \iff \sum_{n=1}^N A^{(n)} = 0.$$

By a classical Marsden-Weinstein theorem (see [5, 4, 6]) the factor with respect to the action of corresponding group of a level of momentum is a symplectic space, let us denote it by  $M$ :

$$M := \tilde{M}|_{\Sigma=0} / \text{GL}(m) = \mathcal{O}(\chi_1) \times \cdots \times \mathcal{O}(\chi_N)|_{\Sigma=0} / \text{GL}(m).$$

A point of  $M$  we denote by  $((A^{(\bar{n})}))$ , it is an equivalence class of sets  $A^{(\bar{n})}$ :

$$A^{(\bar{n})} \sim A'^{(\bar{n})} \iff \exists g \in \text{GL}(m) : A^{(\bar{n})} = g^{-1} A'^{(\bar{n})} g$$

The goal of this paper is to construct explicitly a symplectic birational isomorphism between  $M$  and a *standard* symplectic space, that is a Cartesian product of the orbits  $\mathcal{O}(\chi)$ .

## 2. SUBSET $\tilde{D} \xrightarrow{\pi} D$ OF $\tilde{M}|_{\Sigma=0} \xrightarrow{\pi} M$

Let us consider a fiber bundle  $\tilde{M}|_{\Sigma=0} \xrightarrow{\pi} M$ . The Hamiltonian reduction theory states that a form  $\tilde{\omega}|_{\Sigma=0}$ , that is a restriction of  $\tilde{\omega}$  on the submanifold  $\sum A^{(n)} = 0$ , takes the same values on all vectors  $\tilde{\xi} \in T\tilde{M}$  with the same projection on  $TM$ :

$$\pi\tilde{\xi}_1 = \pi\tilde{\xi}_2 := \xi, \quad \pi\tilde{\eta}_1 = \pi\tilde{\eta}_2 := \eta \implies \tilde{\omega}(\tilde{\xi}_1, \tilde{\eta}_1) = \tilde{\omega}(\tilde{\xi}_2, \tilde{\eta}_2) =: \omega(\xi, \eta).$$

It means that  $\tilde{\omega}$  is in the image of  $\pi^*$ , of the pullback of  $\pi$ . It implies that on the image of  $\pi$  the form  $\omega$  is well defined by an equality  $\tilde{\omega} = \pi^*\omega$ . The Hamiltonian reduction theory guarantees that  $\omega$  is symplectic, i.e. non-degenerate and closed.

A regular way to construct (local coordinate) functions on the factor-manifold  $M$  is to specify a (local) section  $\sigma_D$  of the bundle

$$\sigma_D : D \rightarrow \tilde{M}|_{\Sigma=0}, \quad \pi\sigma_D = id, \quad \text{where } D \subset M,$$

and restrict on the  $\sigma_D(D) \subset \tilde{M}$  functions defined on  $\tilde{M}$ , in particular – matrix elements  $A_{ij}^{(n)}$ , they form a standard set of coordinate functions on  $\tilde{M}$ .

Practically we specify a domain  $\tilde{D} \subset \tilde{M}$ , and restrict the fibre bundle  $\tilde{M}|_{\Sigma=0} \xrightarrow{\pi} M$  on it:  $\tilde{D} \xrightarrow{\pi} D := \pi(\tilde{D})$ . There are rational sections of this bundle. We define such a section as a graph, a subset  $\sigma_D(D) \subset \tilde{D} \subset \tilde{M}|_{\Sigma=0}$  that intersects all fibres of  $\tilde{D} \xrightarrow{\pi} D$  exactly once. The Definition-Notation1 below is a definition-description of these submanifolds. Let us realize this programm.

**Definition–Notation 1.** By  $\tilde{D} := \tilde{D}(N, N-1, N-2)$  we denote such a subset of  $\tilde{M}|_{\Sigma=0} \supset \tilde{D}$  that for sets  $A^{(\bar{n})}$  from it there exists such a  $g \in \text{GL}(m)$  that

- (1)  $g^{-1}A^{(N-1)}g$  is upper-triangular,
- (2)  $g^{-1}A^{(N-2)}g$  is lower-triangular
- (3)  $\sum_{k=1}^m (g^{-1}A^{(N)}g)_{ik} = \sum_{k=1}^m (g^{-1}A^{(N)}g)_{jk} \quad \forall i, j.$

To discuss these three conditions, consider an action of  $GL(m)$  on the set of frames of  $\mathbb{C}^m$ . There is one-to-one correspondence (faithful representation) between changes of a fixed basis  $(\mathfrak{E}) \stackrel{def}{=} (\vec{e}_1, \dots, \vec{e}_m)$  of  $\mathbb{C}^m$ , and elements  $g$  of  $GL(m)$ :  $(\mathfrak{E}) \rightarrow (\mathfrak{E}') = (\mathfrak{E})g$ . Elements of algebra  $gl(m)$  can be considered as linear transformations of this  $\mathbb{C}^m$ ; changes of the basis of  $\mathbb{C}^m$  induce the adjoint representation of the group on its algebra by similarity transformations.

In this interpretation Definition-Notation 1 means that there is such a basis  $(\mathfrak{E})$  of  $\mathbb{C}^m$  that three specified matrices  $A^{(N-1)}$ ,  $A^{(N-2)}$  and  $A^{(N)}$  have special forms. Consider  $A^{(N-1)}$ ,  $A^{(N-2)}$ ,  $A^{(N)}$  in the basis  $(\mathfrak{E})$ .

First of all, let us notice that  $A^{(N-1)}$  and  $A^{(N-2)}$  are triangular, consequently their diagonal elements are eigenvalues. The specification of the basis with necessary properties (1),(2) gives some ordering of their eigenvalues  $\lambda_1^{(N-1)}, \dots, \lambda_m^{(N-1)} =: \lambda_{\vec{k}}^{(N-1)}$ ,  $\lambda_1^{(N-2)}, \dots, \lambda_m^{(N-2)} =: \lambda_{\vec{k}}^{(N-2)}$ .

Let us consider the last property (3). It means that one of eigenvectors of  $A^{(N)}$  has all the components equal to each other, let us add its eigenvalue (denote it by  $\lambda^{(N)}$ ) that is a common value of the sums in (3), to the set of discrete parameters – the orders of eigenvalues.

Let us denote by  $(\mathfrak{E})(\lambda^{(N)}; \lambda_{\vec{k}}^{(N-1)}, \lambda_{\vec{k}}^{(N-2)})$  the basis in which diagonal elements of  $A^{(N-1)}$ ,  $A^{(N-2)}$  have preassigned orders of eigenvalues, and the sum of every row of  $A^{(N)}$  is equal to  $\lambda^{(N)}$  that is the assigned eigenvalue of  $A^{(N)}$ .

These orderings are the discrete parameters of our construction. In a generic case we can choose  $\lambda^{(N)}$  and orderings of eigenvalues of  $A^{(N-1)}$ ,  $A^{(N-2)}$  in an arbitrary way, in special points of  $\tilde{D}$  some combinations of  $\lambda$ 's can not be realized, but the defining property of  $\tilde{D}$  is the following: *there is at least one such a basis  $(\mathfrak{E})$ .*

### 3. UPPER- AND LOWER- NORMAL JORDAN FORMS RELATED TO THE ASSIGNED ORDERING OF THE DIAGONAL ELEMENTS

Let us fix parameters  $\lambda^{(N)}, \lambda_{\vec{k}}^{(N-1)}, \lambda_{\vec{k}}^{(N-2)}$  in some a way and consider a subdomain  $\tilde{D}(N, N-1, N-2; \lambda^{(N)}; \lambda_{\vec{k}}^{(N-1)}, \lambda_{\vec{k}}^{(N-2)}) \subset \tilde{D}(N, N-1, N-2)$ , where the basis  $(\mathfrak{E})(\lambda^{(N)}; \lambda_{\vec{k}}^{(N-1)}, \lambda_{\vec{k}}^{(N-2)})$  exists.

We say that matrix  $A_{J+}^{(N-1)}$  has upper-Jordan form *related to the ordering  $\lambda_{\vec{k}}^{(N-1)}$  of its eigenvalues*, if it is upper-triangular with assigned order of the diagonal entries; all non-diagonal entries are zero except units that correspond to the Jordan chains of generalized eigenvectors.

The only difference from the standard Jordan normal form is that the eigenvectors and the generalized eigenvectors that form the basis of  $\mathbb{C}^m$  in which the matrix has the standard Jordan form are rearranged in the special order in accordance with the given order of the diagonal elements.

In the same way we define a lower-Jordan form  $A_{J-}^{(N-2)}$  *related to the ordering  $\lambda_{\vec{k}}^{(N-2)}$  of the eigenvalues* of  $A^{(N-2)}$ .

The difference between upper- and lower- forms in the directions of Jordan chains.

The following simple fact from the matrix theory is very important for us.

**Proposition 1.** *Any upper-(lower-)triangular matrix can be transformed into the upper-(lower-)Jordan normal form with the same diagonal via similarity transformation by some upper-(lower-)triangular matrix.*

*If all the eigenspaces of the matrix are one-dimensional (Restriction 1), all such similarity transformations differ from each other by upper-(lower-)triangular factor, that is an arbitrary diagonal for the diagonalizable matrix.*

*Proof.* The first statement is evident. The second one is true because if we have only one eigenvector corresponding to the eigenvalue, the only freedom in the procedure of constructing of Jordan basis is a choice of the lead vector of the Jordan chain. It gives the triangular transformation of the set of base vectors in the invariant subspace corresponding to this eigenvalue.  $\square$

#### 4. GAUSS DECOMPOSITION AND FUNCTION $(\mathfrak{E})(\cdot)$

Let  $A$  and  $B$  be any matrices. Let us denote a basis in which  $A$  has upper-Jordan form related to a fixed ordering of eigenvalues by  $(\mathfrak{E}_+)$ , and let us denote a basis in which  $B$  has lower-Jordan form related to a fixed ordering of eigenvalues by  $(\mathfrak{E}_-)$ . Let us denote an element of  $\text{GL}(m)$  that transforms  $(\mathfrak{E}_+)$  to  $(\mathfrak{E}_-)$  by  $\Phi_{+-}$ :

$$(\mathfrak{E}_+)\Phi_{+-} = (\mathfrak{E}_-), \quad \Phi_{+-} \in \text{GL}(m).$$

**Theorem 1.** *The following two statements are equivalent:*

- I. *Matrix  $\Phi_{+-}$  admits the Gauss decomposition*

$$\Phi_{+-} = \Phi_+ \Phi_-$$

*on the upper- and lower- triangular  $\Phi_+$  and  $\Phi_-$ .*

- II. *There is such a basis  $(\mathfrak{E})$ , in which  $A$  is upper-triangular and  $B$  is lower-triangular.*

*Proof.* If  $\Phi_+ \Phi_-$  is the Gauss decomposition of  $\Phi_{+-}$ , the desired basis  $(\mathfrak{E})$  is:

$$(\mathfrak{E}) = (\mathfrak{E}_+)\Phi_+ = (\mathfrak{E}_-)\Phi_-^{-1},$$

so “I” implies “II”.

Otherwise, let  $(\mathfrak{E})$  be a basis in which  $A$  is upper-triangular,  $B$  is lower-triangular. A transformation of an upper-(lower-)triangular matrix to the corresponding upper-(lower-)Jordan form can be made by a triangular matrix.

Let us denote the triangular matrices that transform  $A$  and  $B$  to the corresponding (upper- and lower-) normal Jordan form by  $\Phi_+^{-1}$  and  $\Phi_-$ . They are the matrices that give us the Gauss decomposition of the transformation connecting  $(\mathfrak{E}_+)$  and  $(\mathfrak{E}_-)$ .  $\square$

**Theorem 2.** *Let  $A$  and  $B$  be any matrices without two-dimensional eigenspaces, and let condition “II” of the Theorem 1 fulfil. Then*

*$(\mathfrak{E})$  is uniquely defined up to an arbitrary diagonal transformation  $\delta$ :  $(\mathfrak{E}) \sim (\mathfrak{E})\delta$ .*

In other words the directions of the vectors of basis in question are uniquely defined.

*Proof.* Upper-(lower-) Jordan normal form commutes with matrices of a special type only, that is triangular in the case when only one Jordan block corresponds to any eigenvalue (see [3]). That is our case, consequently bases  $(\mathfrak{E}_\pm)$  are defined up

to the corresponding (upper- and lower-) triangular transformations  $\delta_{\pm}^{\Delta}$  (of special kind):  $(\mathfrak{E}'_+) = (\mathfrak{E}_+)\delta_+^{\Delta}$ ,  $(\mathfrak{E}'_-) = (\mathfrak{E}_-)\delta_-^{\Delta}$ .

Gauss decomposition of a given  $\Phi_{+-}$  is unique up to a diagonal factor. The ambiguity of the choices of  $(\mathfrak{E}_{\pm}) = (\mathfrak{E}'_{\pm})\delta_{\pm}^{\Delta}$  gives the following ambiguity of  $\Phi_{+-}$ :

$$\Phi'_{+-} = \delta_+^{\Delta}\Phi_{+-}(\delta_-^{\Delta})^{-1},$$

it has evident decomposition on the product of triangular factors:

$$\Phi'_{+-} = \delta_+^{\Delta}\Phi_{+-}(\delta_-^{\Delta})^{-1} = (\delta_+^{\Delta}\Phi_+)(\Phi_-(\delta_-^{\Delta})^{-1}) =: \Phi'_+\Phi'_-.$$

By the uniqueness of the Gauss decomposition  $\Phi'_+$  and  $\Phi'_-$  are defined up to diagonal factors – right for  $\Phi'_+$  and left for  $\Phi'_-$ , that is the source of diagonal ambiguity of  $(\mathfrak{E})$ .

The ambiguity of  $(\mathfrak{E}_{\pm})$ :  $(\mathfrak{E}'_{\pm}) = (\mathfrak{E}_{\pm})(\delta_{\pm}^{\Delta})^{\pm 1}$  does not effect on the basis  $(\mathfrak{E})$ :

$$(\mathfrak{E}) = (\mathfrak{E}_+)\Phi_+ = (\mathfrak{E}'_+)\delta_+^{\Delta}\Phi_+ = (\mathfrak{E}'_+)\Phi'_+ = (\mathfrak{E}),$$

it proves the present Theorem.  $\square$

Consider the Definition-Notation 1. It states that for the sets  $A^{(\bar{n})}$  from the domain  $\tilde{D}$  it is possible to choose such a basis  $(\mathfrak{E})$  that  $A^{(N-1)}$  will be upper-triangular, and  $A^{(N-2)}$  will be lower-triangular. By the Theorem 2 basis  $(\mathfrak{E})$  is determined the up to the right diagonal factor.

Let us consider the third condition of Definition-Notation 1. It is equivalent to the equality  $A^{(N)}\vec{1} = \lambda^{(N)}\vec{1}$ , where  $\vec{1}$  is a column of units, and  $\lambda^{(N)}$  is a common value of the sums. Changes of basis change coordinates of eigenvectors:  $(\mathfrak{E}) = (\mathfrak{E}')\delta \rightarrow \vec{f} = \delta\vec{f}'$ . Consequently the conditions of the Definition-Notation 1 determine the basis  $(\mathfrak{E})$  up to a common scalar factor. It is the base of our construction in a direct and a figurative sense. Let us reformulate it.

Let us denote by  $\mathbf{P}$ (“bases of  $\mathbb{C}^m$ ”) the projectivization of the set of frames. Different representatives of one class are bases connected by a scalar factor. In any basis from one equivalence class all matrices  $A$  have the same matrix elements. We have constructed the sections of a projectivised frame-bundle over  $\tilde{D}$ , that are the single-valued functions  $(\mathfrak{E})$  with values in  $\mathbf{P}$ (“bases of  $\mathbb{C}^m$ ”). Let us denote the set of data  $(N, N-1, N-2; \lambda^{(N)}; \lambda_k^{(N-1)}, \lambda_k^{(N-2)})$  by  $(\cdot)$  for short.

**Theorem 3.** *Function  $(\mathfrak{E})(\cdot): \tilde{D}(\cdot) \rightarrow \mathbf{P}$ (“bases of  $\mathbb{C}^m$ ”) is well defined, and all the components  $(\mathfrak{E})_i^j$  of the vectors of the basis  $(\mathfrak{E})$  are rational functions of matrix elements  $A_{ij}^{(n)}$ ,  $n = N, N-1, N-2; i, j, k = 1, \dots, m$ .*

*Proof.* Determining eigenvectors and generalized eigenvectors by a given Jordan normal form is a problem of solving the systems of linear equations, the same for Gauss decomposing of matrices collected from these eigenvectors and generalized eigenvectors. These are rational operations.  $\square$

**Theorem 4.**  *$\tilde{D}(\cdot)$  and  $D(\cdot)$  are open sets in a Zariski topology.*

*Proof.* A complement to  $\tilde{D}$  is defined by a system of algebraic equations – vanishing of upper-left minors of  $\Phi_{+-}$ , that is the condition of the impossibility of Gauss decomposing (see [3]), and vanishing of any component of the eigenvector  $\vec{f}$  of  $A^{(N)}$ .

Complement to  $D$  is defined by the same equations, because the conditions of the Definition-Notation 1 are invariant with respect to the action of  $\text{GL}(m)$ .  $\square$

Let us restrict the fibre bundle  $\tilde{M}|_{\Sigma=0} \xrightarrow{\pi} M$  on  $\tilde{D}(\cdot) \subset \tilde{M}|_{\Sigma=0}$ :  $\tilde{D}(\cdot) \xrightarrow{\pi} D(\cdot)$ . From now on we somehow fix the parameters  $(N, N-1, N-2; \lambda^{(N)}; \lambda_k^{(N-1)}, \lambda_k^{(N-2)})$  and will not write  $(\cdot)$  any more.

**Proposition 2.**  $\tilde{\omega}|_D = \pi^* \omega|_D$ , and 2-form  $\omega|_D$  is a symplectic form on  $D$

*Proof.* Making the restriction we exclude from the consideration only several submanifolds of smaller dimensions.  $\square$

We have constructed the function  $(\mathfrak{E}) : D \rightarrow \mathbf{P}$ (“ bases of  $\mathbb{C}^m$  ”), that induces a rational section  $\sigma$  of fibre-bundle  $\tilde{D} \xrightarrow{\pi} D$ :

$$\sigma( ((A^{(\bar{n})})) ) = A_\sigma^{(1)}, \dots, A_\sigma^{(N)} = A_\sigma^{(\bar{n})},$$

where  $A_\sigma^{(\bar{n})}$  is a representative of  $((A^{(\bar{n})}))$  calculated in the basis  $(\mathfrak{E})$ . In other words, it is such a representative of the conjugacy class that has the triangular forms of  $A^{(N-1)}, A^{(N-2)}$  with assigned orders of the diagonal elements and the assigned value of sums of all rows of  $A^N$ .

Section  $\sigma$  depends on discrete parameters:  $N, N-1, N-2; \lambda^{(N)}; \lambda_k^{(N-1)}, \lambda_k^{(N-2)}$ , but we will not write them again.

#### 5. PROJECTION $\hat{\pi}_{\lambda^{(N)}}^{(N)}$ ON THE ORBIT $\mathcal{O}$ OF THE SMALLER DIMENSION

Let us denote a natural projections on the Cartesian factors  $\mathcal{O}(\chi_n)$  by  $\pi^{(n)}$ . These projections are defined for all points of  $\mathfrak{gl}(m) \times \dots \times \mathfrak{gl}(m)$ :

$$\pi^{(n)} : A^{(1)}, \dots, A^{(N)} \longrightarrow A^{(n)} \in \mathcal{O}(\chi_n).$$

Let us consider a submanifold  $\sigma(D) \subset \tilde{D}$ . For its points we will define a projection  $\hat{\pi}_{\lambda^{(N)}}^{(N)} : \mathcal{O}(\chi_N) \rightarrow \mathcal{O}(\chi_N/\lambda - \lambda^N)$ , where the orbit  $\mathcal{O}(\chi_N/\lambda - \lambda^N)$  corresponds to the polynomial  $\chi_N/\lambda - \lambda^N$ . Its degree is less than the degree of  $\chi_N$  by a unit:  $\mathcal{O}(\chi_N/\lambda - \lambda^N) \subset \mathfrak{gl}(m-1)$ .

Let us do it. All matrices from  $\mathcal{O}(\chi_N) \cap \sigma(D)$  have a fixed eigenvector  $\vec{1}$ , we take it as an  $m$ 'th basis vectors of  $\mathbb{C}^m$ :  $(\hat{\mathfrak{E}}) := (\mathfrak{E})\Xi$ , where  $\Xi$  is the following constant matrix

$$\Xi := \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 1 \\ 0 & 1 & \dots & 0 & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 & 1 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}, \quad \Xi^{-1} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & -1 \\ 0 & 1 & \dots & 0 & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & -1 \\ 0 & 0 & \dots & 0 & 1 & -1 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}$$

In the basis  $(\hat{\mathfrak{E}})$  matrix  $A^{(N)}$  takes a block-triangular form:

$$(1) \quad \Xi^{-1} A^{(N)} \Xi =: \begin{pmatrix} \hat{A} & \vec{0} \\ a^T & \lambda^{(N)} \end{pmatrix},$$

where by  $\hat{A}$  we denote a matrix whose characteristic polynomial is  $\chi_N/\lambda - \lambda^{(N)}$ ;  $a^T$  is an  $m-1$ -dimensional vector-row, it is the last row of  $A^{(N)}$  without the  $m$ 's element:  $a_k^T = A_{mk}^{(N)}$ ,  $k \neq m$ .

**Definition–Notation 2.** Let us denote by  $\hat{\pi}_{\lambda_N}^{(N)}$  a map

$$\hat{\pi}_{\lambda_N}^{(N)} : A^{(1)}, \dots, A^{(N)} \longrightarrow \hat{A},$$

where  $\hat{A}$  is  $m-1 \times m-1$  upper-left-corner block of  $\Xi^{-1}A^{(N)}\Xi$ .

**Proposition 3.**  $\hat{\pi}_{\lambda_N}^{(N)}(A^{(\bar{n})})$  is a projection on the orbit  $\mathcal{O}(\chi_N/\lambda - \lambda^N)$ .

*Proof.* By a construction the characteristic polynomial of  $\hat{A}$  is  $\chi_N/\lambda - \lambda^N$ , so it is sufficient to prove that all eigenspaces of  $\hat{A}$  are one-dimensional.

Let  $\lambda^{(N)} = 0$ . It is not a restriction because we can subtract  $\lambda^{(N)}\mathbf{I}$  from  $A^{(N)}$ , it will not change the dimensions of the eigenspaces of the block. Assume that there are two different (not proportional) eigenvectors  $f_1, f_2$  of  $\hat{A}$  that correspond to one  $\lambda$ .

If  $\lambda \neq 0$ , vectors  $(f_i, \langle a^T f_i \rangle / \lambda)^T$ ,  $i = 1, 2$  are two different eigenvectors of  $\Xi^{-1}A^{(N)}\Xi$  corresponding to one  $\lambda$  that contradicts Restriction 1. Let  $\lambda = 0$ . If, say,  $\langle a^T f_1 \rangle = 0$  then  $(f_1, 0)^T$  is the eigenvector of  $\Xi^{-1}A^{(N)}\Xi$  in addition to  $(\vec{0}, 1)^T$ . If  $\langle a^T f_1 \rangle \neq 0 \neq \langle a^T f_2 \rangle$ , the second eigenvector corresponding to  $\lambda = 0$  is  $(\langle a^T f_2 \rangle f_1 - \langle a^T f_1 \rangle f_2, 0)$ . It is not a zero because of the linear independence of  $f_i$ .  $\square$

**Definition–Notation 3.** Let us denote by  $\hat{\pi}$  a following projection of  $\sigma(D)$  on the Cartesian product of the orbits:  $\sigma(D) \longrightarrow \mathcal{O}(\chi_N/\lambda - \lambda^{(N)}) \times \mathcal{O}(\chi_1) \times \dots \times \mathcal{O}(\chi_{N-3})$ ,

$$\hat{\pi}(A^{(\bar{n})}) := \hat{\pi}_{\lambda_N}^{(N)}(A^{(\bar{n})}), \pi^{(1)}(A^{(\bar{n})}), \dots, \pi^{(N-3)}(A^{(\bar{n})}).$$

The product of the orbits is a symplectic space, consequently we construct the rational map between two symplectic spaces. The first space is  $D$ , that is an open dense domain of  $M := \mathcal{O}(\chi_1) \times \dots \times \mathcal{O}(\chi_N)|_{\Sigma=0} / \text{GL}(m) \supset D$ , and the second space is the Cartesian product of the orbits  $\mathcal{O}$ . The main theorem of the present article is

**Theorem 5.** The map  $\hat{\pi}\sigma$  is a symplectic birational isomorphism; in a case of diagonalizable  $A^{(n)}$ 's, that is a case of a general position,  $\hat{\pi}\sigma$  is a symplectic isomorphism that is birational between  $D$  and the product of the orbits.

Before starting the proof let me remind that by the Hamiltonian reduction theory the  $\tilde{\omega}|_{\Sigma=0}$  on  $\tilde{M}|_{\Sigma=0}$  is degenerate and its value is the same on all vectors with the same projection on  $M$ . We can reformulate this statement in the following way:

Let us consider any local section that is a (local) isomorphism on its image:  $\sigma' : M \rightarrow \tilde{M}|_{\Sigma=0}$ ,  $\pi\sigma' = id$ . Let us restrict  $\tilde{\omega}|_{\Sigma=0}$  on the image. We get  $(\tilde{\omega}|_{\Sigma=0})|_{\sigma'(M)} := \tilde{\tilde{\omega}}$ ;  $\sigma'$  is the (local) isomorphism, consequently its inverse  $\sigma'^{-1}$  exists and define a desired 2-form  $(\sigma'^{-1})^*\tilde{\tilde{\omega}}$  on (a domain of)  $M$ , that is the pullback of  $\tilde{\tilde{\omega}}$  by the  $\sigma'^{-1}$ .

Generically this form (of cause!) depends on the choice of the section  $\sigma'$ . It is the main result of the theory of the Hamiltonian reduction — to specify the conditions (a Poisson action of a group, level of a momentum map etc.) in such a way that 2-form  $(\sigma'^{-1})^*\tilde{\tilde{\omega}}$  would not depend on the choice of the section  $\sigma'$ . It is our freedom that we can choose  $\sigma'$  as we like. It follows from the presented speculations that the value of the sum

$$(2) \quad \omega = \omega_1 + \dots + \omega_N$$



does *not* depend on the choice of the submanifold  $\sigma(D)$  on which we restrict the sum. We choose it *in the most comfortable* way – actually it is the Definition-Notation 1. Let us prove the Theorem now.

*Proof.* Consider the projection  $\hat{\pi}_{\lambda_N}^{(N)}$  of the  $\mathcal{O}(\chi_N)$  on  $\mathcal{O}(\chi_N/\lambda - \lambda_N)$ . Let us denote by  $\hat{\pi}_{\lambda_N}^{(N)*}\hat{\omega}$  the pullback of the standard symplectic form  $\hat{\omega}$  on the orbit  $\mathcal{O}(\chi_N/\lambda - \lambda_N)$  by this projection.

We have two forms defined on  $\mathcal{O}(\chi_N)$ , the standard symplectic form  $\omega_N$  and the pullback in question. They can not coincide because  $\omega_N$  is symplectic but the pullback is degenerate – the dimension of  $\mathcal{O}(\chi_N/\lambda - \lambda_N)$  is smaller than the dimension of  $\mathcal{O}(\chi_N)$ .

Let us consider the projection  $\hat{\pi}_{\lambda_N}^{(N)}$  of  $D$  on  $\mathcal{O}(\chi_N)$ . It is not surjective. Its image is a submanifold  $\hat{\pi}_{\lambda_N}^{(N)}(D)$  of a smaller dimension. Consider the two mentioned forms *restricted on the submanifold*  $\hat{\pi}_{\lambda_N}^{(N)}(D) \subset \mathcal{O}(\chi_N)$ .

**Lemma 5.1.**

$$(\hat{\pi}_{\lambda_N}^{(N)*}\hat{\omega})|_{\hat{\pi}_{\lambda_N}^{(N)}(D)} = (\omega_N)|_{\hat{\pi}_{\lambda_N}^{(N)}(D)}$$

*Proof.* Let us print the expression [7] for the standard Lie-Poisson form on an orbit:

$$\omega_{LP}(\xi, \eta) = -\text{tr } U_\xi \dot{A}_\eta = \text{tr } U_\eta \dot{A}_\xi,$$

where  $\xi, \eta \in T_A \mathcal{O}$  are two tangent vectors to an orbit  $\mathcal{O}$  in a point  $A \in \mathcal{O}$ :

$$\xi = [A, U_\xi] = \left. \frac{d}{dt} \right|_{t=0} A_\xi(t) =: \dot{A}_\xi, \quad \eta = [A, U_\eta] = \left. \frac{d}{dt} \right|_{t=0} A_\eta(t) =: \dot{A}_\eta, \quad A_\xi(0) = A_\eta(0) = A.$$

All matrices  $A^{(N)}$  in the special basis  $(\mathfrak{E})$  (actually they form the  $\hat{\pi}_{\lambda_N}^{(N)}(D)$ ) have the constant eigenvector  $\vec{1}$ , and after the similarity transformation by the constant matrix  $\Xi$  take the block-triangle form (1). The printed expression for the Lie-Poisson form is invariant with respect to the constant similarity transformations and having been calculated for the block-diagonal matrix with the fixed diagonal element  $\lambda^{(N)}$  gives the desired — it is just the same expression, where we should write the first  $m-1 \times m-1$  upper-left-diagonal blocks of all matrices  $\dot{A}_\xi, \dot{A}_\eta, U_\xi, U_\eta$  transformed by  $\Xi$ , that is the projection  $\hat{\pi}_{\lambda_N}^{(N)}$ .

Lemma has been proved  $\square$

We have proved that the term  $\omega_N$ , in the sum (2), can be replaced by  $\hat{\pi}_{\lambda_N}^{(N)*}\hat{\omega}$ , i.e. we can calculate the contribution of  $A^{(N)}$  after its projecting on  $\mathcal{O}(\chi_N/\lambda - \lambda^{(N)})$ .

Let us consider the summands  $\omega_{N-1}$  and  $\omega_{N-2}$ . By construction the images  $\pi^{(N-1)}\sigma(A^{(\bar{n})})$  and  $\pi^{(N-2)}\sigma(A^{(\bar{n})})$  belong to Lagrangian submanifolds of triangular matrices, consequently corresponding summands vanish and we get the announced *rational symplectic projection of  $D \subset M$  on the product of the  $N-2$  orbits*.

To finish the proof we should construct a rational inverse transformation.

We have  $N-3$  matrices  $A^{(n)} \in \mathcal{O}(\chi_n)$ , the diagonal entries  $\lambda_{\bar{n}}^{(N-1)}, \lambda_{\bar{n}}^{(N-2)}$  of the triangular  $A^{(N-1)}, A^{(N-1)}$ , eigenvalue  $\lambda^{(N)}$  of  $A^{(N)}$  and matrix  $\hat{A} \in \mathcal{O}(\chi_N/\lambda - \lambda_N)$  as the initial data. We have to reconstruct  $A^{(N)}, A^{(N-1)}, A^{(N-2)}$  using this data and the equation  $\sum_{n=1}^N A^{(n)} = 0$ .

To do this explicitly we introduce some notations

For any matrix  $A \in \text{gl}(m)$  let us denote its projections on the diagonal, under-diagonal and over-diagonal subalgebras by  $A_{=}$ ,  $A_{>}$  and  $A_{<}$ :

$$(A_{=})_{ij} = A_{ii}, \quad i = j, \quad (A_{=})_{ij} = 0, \quad i \neq j$$

$$(A_{>})_{ij} = A_{ij}, \quad i > j, \quad (A_{>})_{ij} = 0, \quad i \leq j,$$

$$(A_{<})_{ij} = A_{ij}, \quad i < j, \quad (A_{<})_{ij} = 0, \quad i \geq j$$

First of all let us construct the diagonal part of  $A^{(N)}$ . All the diagonal entries of  $A^{(n)}$ ,  $n < N$  are given — matrices  $A^{(N-1)}$  and  $A^{(N-2)}$  are diagonal with eigenvalues on their diagonals, consequently we have to put

$$A_{=}^{(N)} := - \sum_{n=1}^{N-1} A_{=}^{(n)}.$$

Now let us construct  $A^{(N)}$ . It is a matrix on the orbit  $\mathcal{O}(\chi_N)$  with given projection  $\hat{A}$  and given diagonal. Matrix  $\Xi$  that defines the projection  $A^{(N)} \rightarrow \hat{A}$  has such simple structure that we can easily write the answer. By the given diagonal we can specify the last row of  $A$  in such a way that  $\hat{A}$  will be the assigned  $m-1 \times m-1$  matrix.

To present an explicit formula for  $A^{(N)}$  let us introduce the operation  $\hat{A} \rightarrow \overset{\circ}{\hat{A}}$ , that makes  $m \times m$  matrix  $\overset{\circ}{\hat{A}}$  from any  $m-1 \times m-1$  matrix  $\hat{A}$  by adding the last zero row and the last zero column. Let us denote this embedding of  $\text{gl}(m-1) \rightarrow \text{gl}(m)$  by a small circle over the name of a matrix:  $\hat{A} \rightarrow \overset{\circ}{\hat{A}}$ .

Some more notations. Let us denote  $\vec{\mathbf{1}} \in \mathbb{C}^m$ ,  $\vec{\mathbf{1}} \otimes \vec{\mathbf{1}} \in \text{gl}(m)$ ,  $(0 \dots 01) \otimes \vec{g} \in \text{gl}(m)$ , where  $\vec{g}$  is any vector:

$$(\vec{\mathbf{1}})_i = 1; \quad (\vec{\mathbf{1}} \otimes \vec{\mathbf{1}})_{ij} = 1 \forall i, j; \quad ((0 \dots 01) \otimes \vec{g})_{ij} = 0, \quad j \neq m, \quad ((0 \dots 01) \otimes \vec{g})_{im} = g_i.$$

It can be easily verified that

$$A^{(N)} = \overset{\circ}{\hat{A}} + (\vec{\mathbf{1}} \otimes \vec{\mathbf{1}})(A_{=}^{(N)} - \overset{\circ}{\hat{A}}_{=}) - (0 \dots 01) \otimes (\overset{\circ}{\hat{A}} \vec{\mathbf{1}}).$$

Here  $(0 \dots 01) \otimes (\overset{\circ}{\hat{A}} \vec{\mathbf{1}})$  is the product of the row  $(0 \dots 01)$  on the column (it is  $\vec{g}$ ) that is the product of  $\overset{\circ}{\hat{A}}$  and the column  $\vec{\mathbf{1}}$ ;  $(A_{=}^{(N)} - \overset{\circ}{\hat{A}}_{=})$  is a diagonal matrix  $m \times m$ .

Matrix  $A^{(N)}$  and diagonal entries of all other matrices are determined now. Finally

$$A_{<}^{(N-1)} := - \sum_{\substack{n=1 \\ n \neq N-1}}^N A_{<}^{(n)}, \quad A_{>}^{(N-2)} := - \sum_{\substack{n=1 \\ n \neq N-2}}^N A_{>}^{(n)}, \quad A_{>}^{(N-1)} = A_{<}^{(N-2)} = 0 \in \text{gl}(m).$$

□

## REFERENCES

- [1] M. V. Babich “O koordinatah na fazovyh prostranstvah sistemy uravnenij Shlezingera i sistemy uravnenij Garn’je–Penleve 6”, Doklady Akademii Nauk, serija Matematika, t. 412, No 4, s. 1-5, (2007) (in Russian);  
English translation: M. V. Babich “About Coordinates on the Phase Spaces of the Schlesinger System and the Garnier-Painlevé 6 System”, Doklady Mathematics, Vol. 75, No. 1, 71 p., (2007). ArXive version:  
<http://arxiv.org/ps/math.SG/0605544>, 2006.
- [2] I.M.Gelfand “Lektzii po linejnoj algebre”, M: Nauka, 272 p., (1971) (in Russian);  
English translation: I. M. Gelfand, “Lectures on linear algebra”, Dover Publications, 185 p., (1989).
- [3] F. R. Gantmacher, “Teorija matritz”, M: Nauka, 576 p., (1966) (in Russian);  
English translation: F. R. Gantmacher, “The theory of matrices”, RI: AMS Chelsea Publishing. x, 374 p., (1998).
- [4] V.I.Arnold “Matemeticheskiye metody klassicheskoy mehaniki”, M: Nauka, 472 p., (1989) (In Russian);  
English translation: V. I. Arnold “Mathematical methods of classical mechanics”, Springer-Verlag. x, 462 p., (1978).
- [5] J. Marsden, A. Weinstein, “Reduction of symplectic manifold with symmetry,” Rep. Mathematical Physics, 5(1): 121-130, 1974.
- [6] A.G. Rejman, M.A. Semenov-Tian-Shansky, “Integrirujemyje Sistemy. Teoretiko-gruppovoj podhod.” – Institut kompjuternyh issledovanij, Moskva – Izhevsk, 352 p., 2003. (In Russian);  
English version: A. G. Reyman and Semenov-Tian-Shansky, “Group-theoretical methods in the theory of finite-dimensional integrable systems”, in Dynamical Systems VII, Editors V.I. Arnold and S.P. Novikov, Encyclopaedia of Mathematical Sciences, V.16, Berlin, Springer, 116-225, 1994.
- [7] N. Hitchin, “Geometrical aspects of Schlesinger’s equation,” Journal of Geometry and Physics, v. 23, 287–300, 1997.

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